DOWNSIDE RISK –
IMPLICATIONS FOR FINANCIAL MANAGEMENT

ROBERT ENGLE

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RISK AND RETURN

- THE TRADE-OFF BETWEEN RISK AND RETURN IS THE CENTRAL PARADIGM OF FINANCE.
- HOW MUCH RISK AM I TAKING?
- HOW SHOULD I RESPOND TO RISKS THAT VARY OVER TIME?
- HOW SHOULD I RESPOND TO RISKS OF VARIOUS MATURITIES?

DOWNSIDE RISK

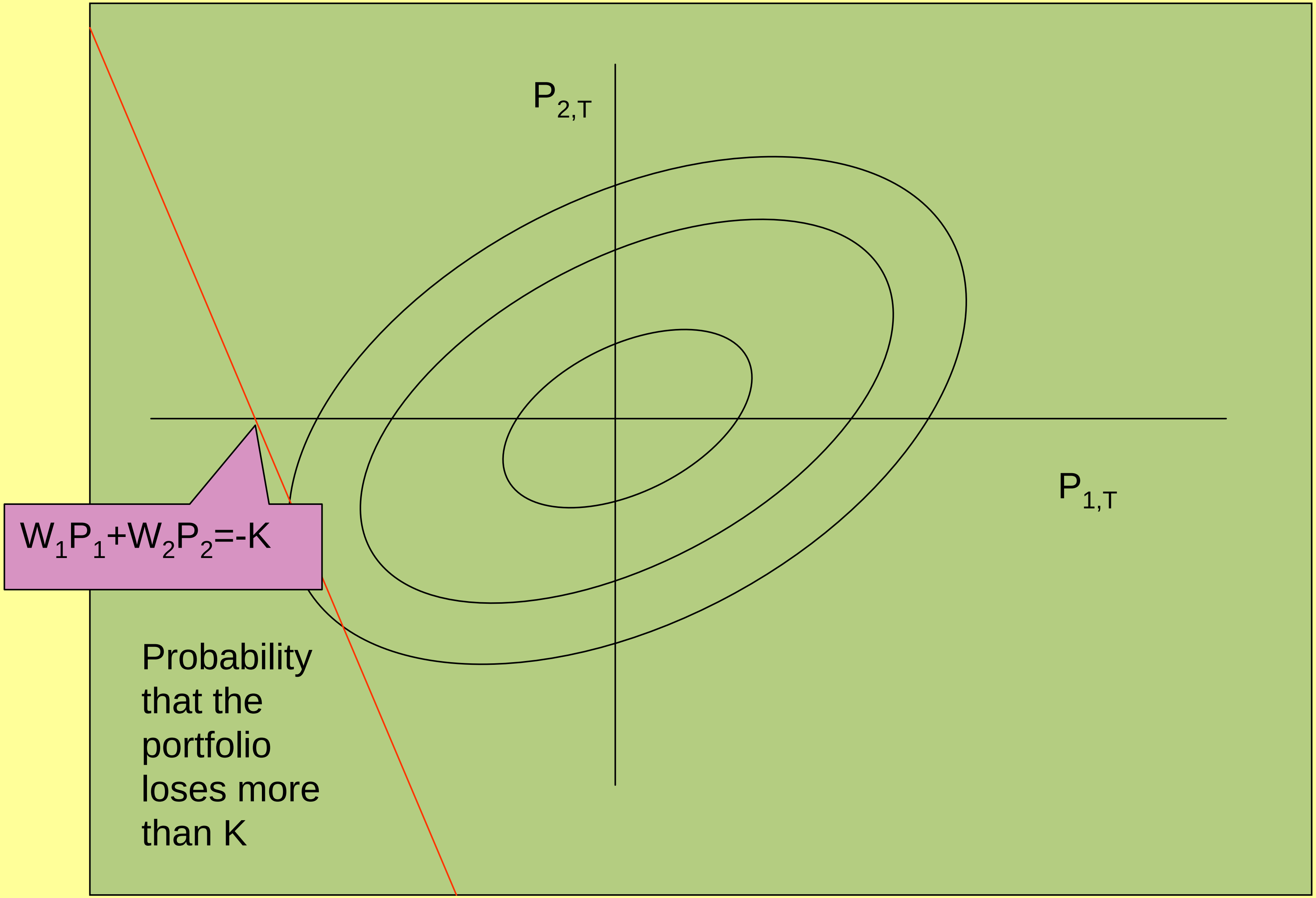
- THE RISK OF A PORTFOLIO IS THAT ITS VALUE WILL DECLINE, NOT THAT IT WILL INCREASE HENCE DOWNSIDE RISK IS NATURAL.
- MANY THEORIES AND MODELS ASSUME SYMMETRY: c.f. MARKOWITZ, TOBIN, SHARPE AND VOLATILITY BASED RISK MANAGEMENT SYSTEMS.
- DO WE MISS ANYTHING IMPORTANT?

MEASURING DOWNSIDE RISK

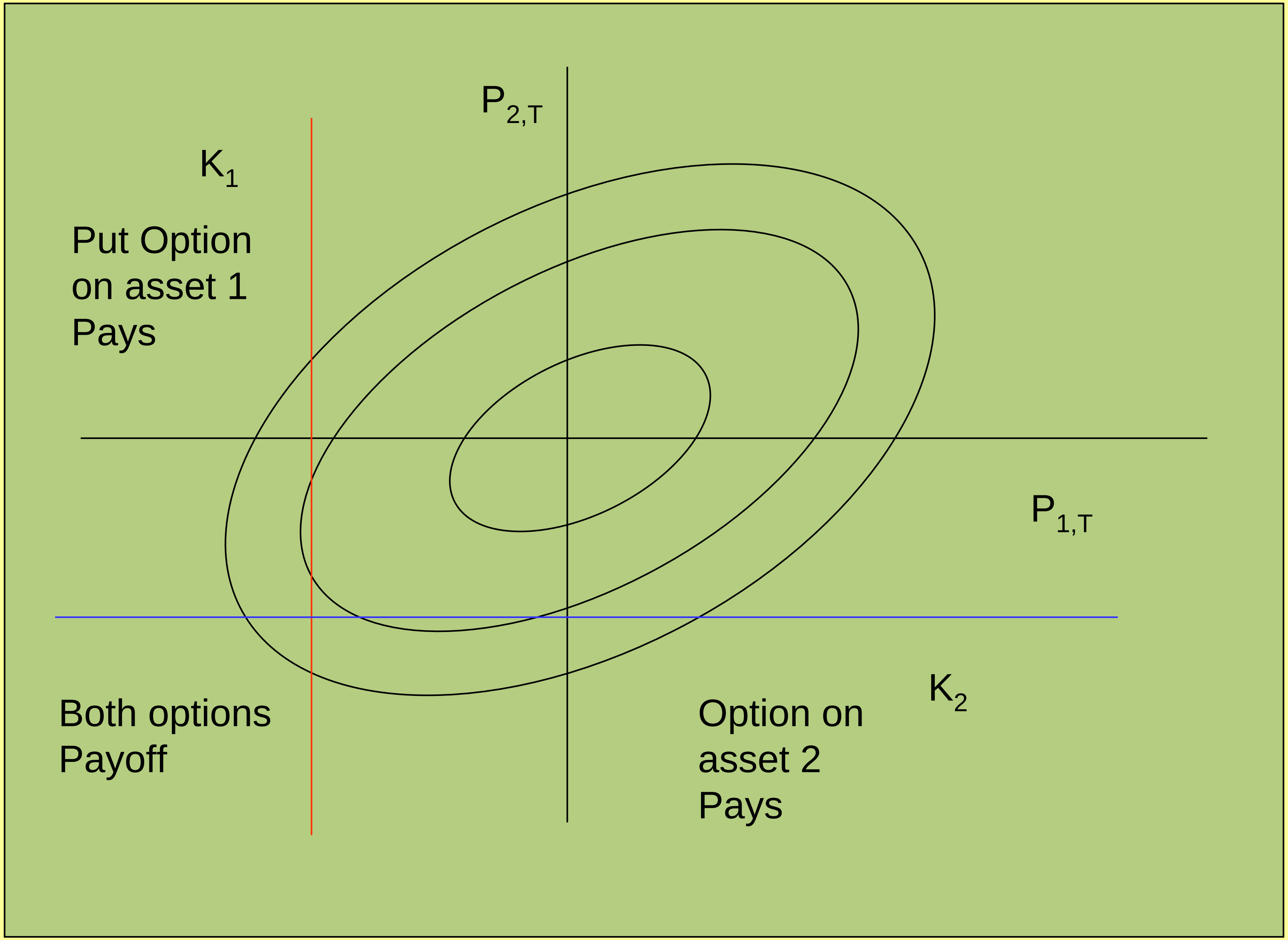
- Many measures have been proposed. Let *r* be the one period continuously compounded return with distribution *f(r)* and mean zero. Let x be a threshold.
  - Skewness $=E\left(r^3\right)/E\left(r^2\right)^{3/2}$
  - Probability of loss $= P\left(r < x\right)$,
  - Expected loss $= E\left(r \middle| r < x\right)$
  - x is the α Value at risk if $P\left(r < -x\right) = \alpha$

MULTIVARIATE DOWNSIDE RISK

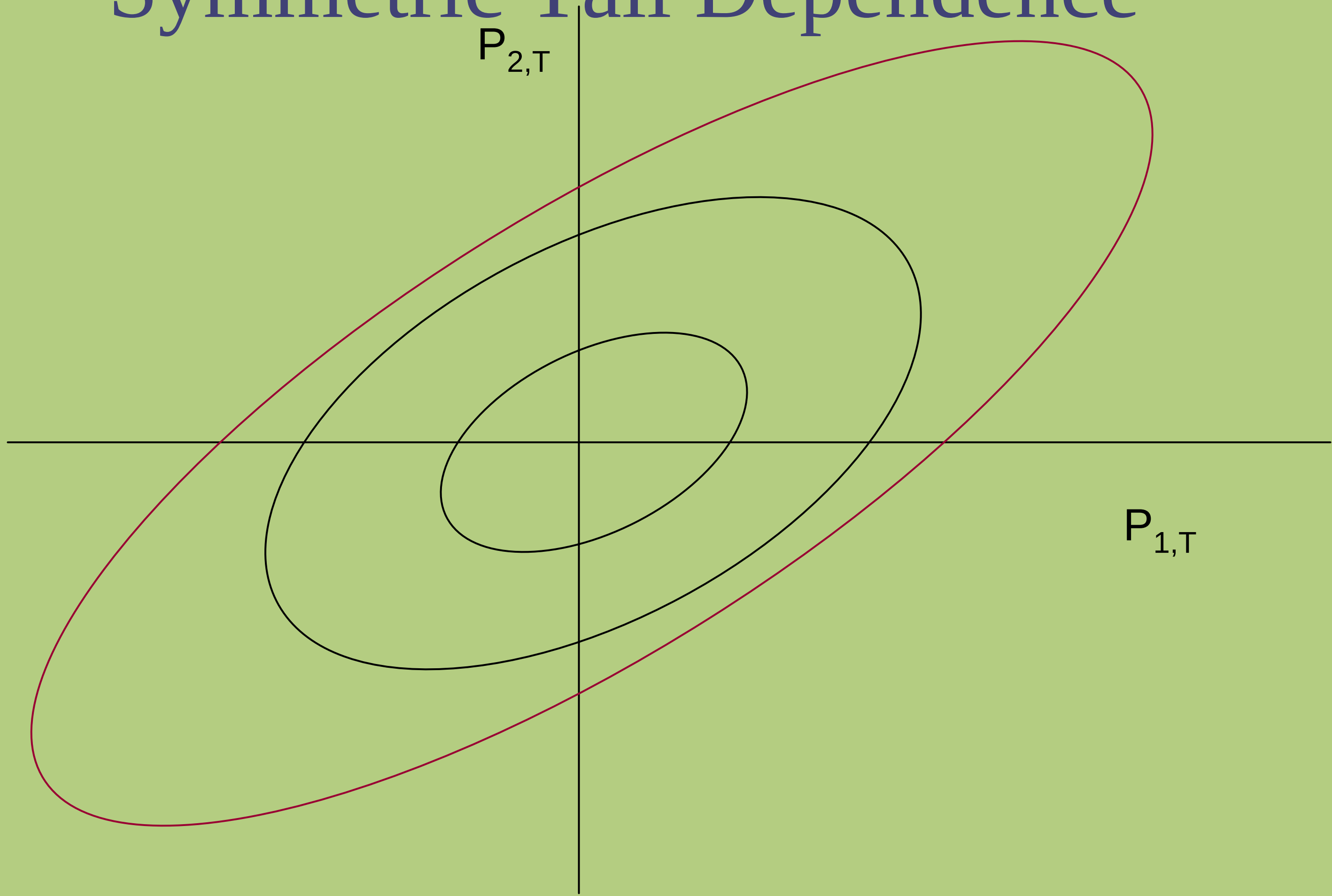
- WHAT IS THE LIKELIHOOD THAT A COLLECTION OF ASSETS WILL ALL DECLINE?
- THIS DEPENDS PARTLY ON CORRELATIONS
- FOR EXTREME MOVES, OTHER MEASURES ARE IMPORTANT TOO.



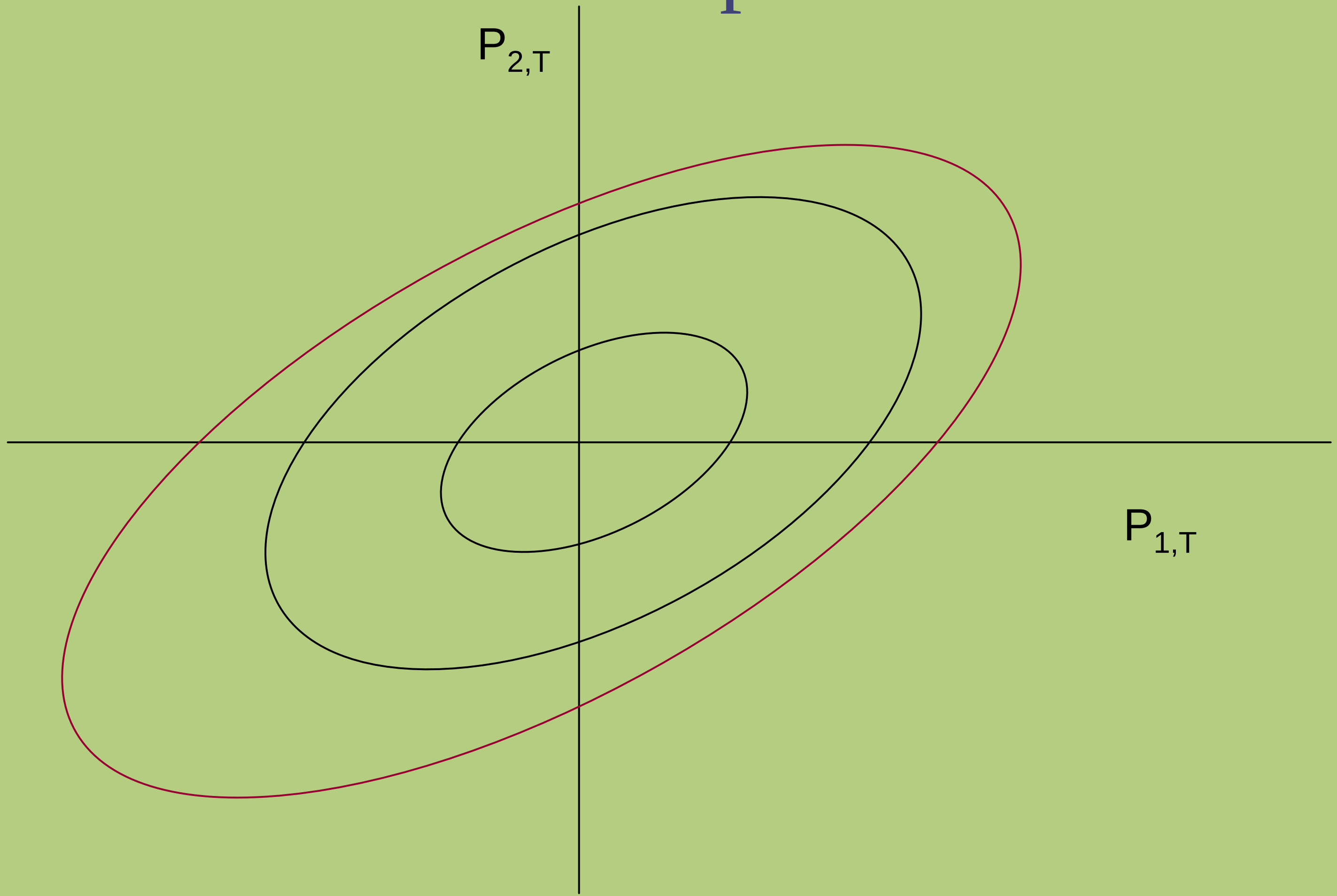
$P_{2,T}$
$P_{1,T}$
$W_1P_1+W_2P_2=-K$
Probability that the portfolio loses more than K

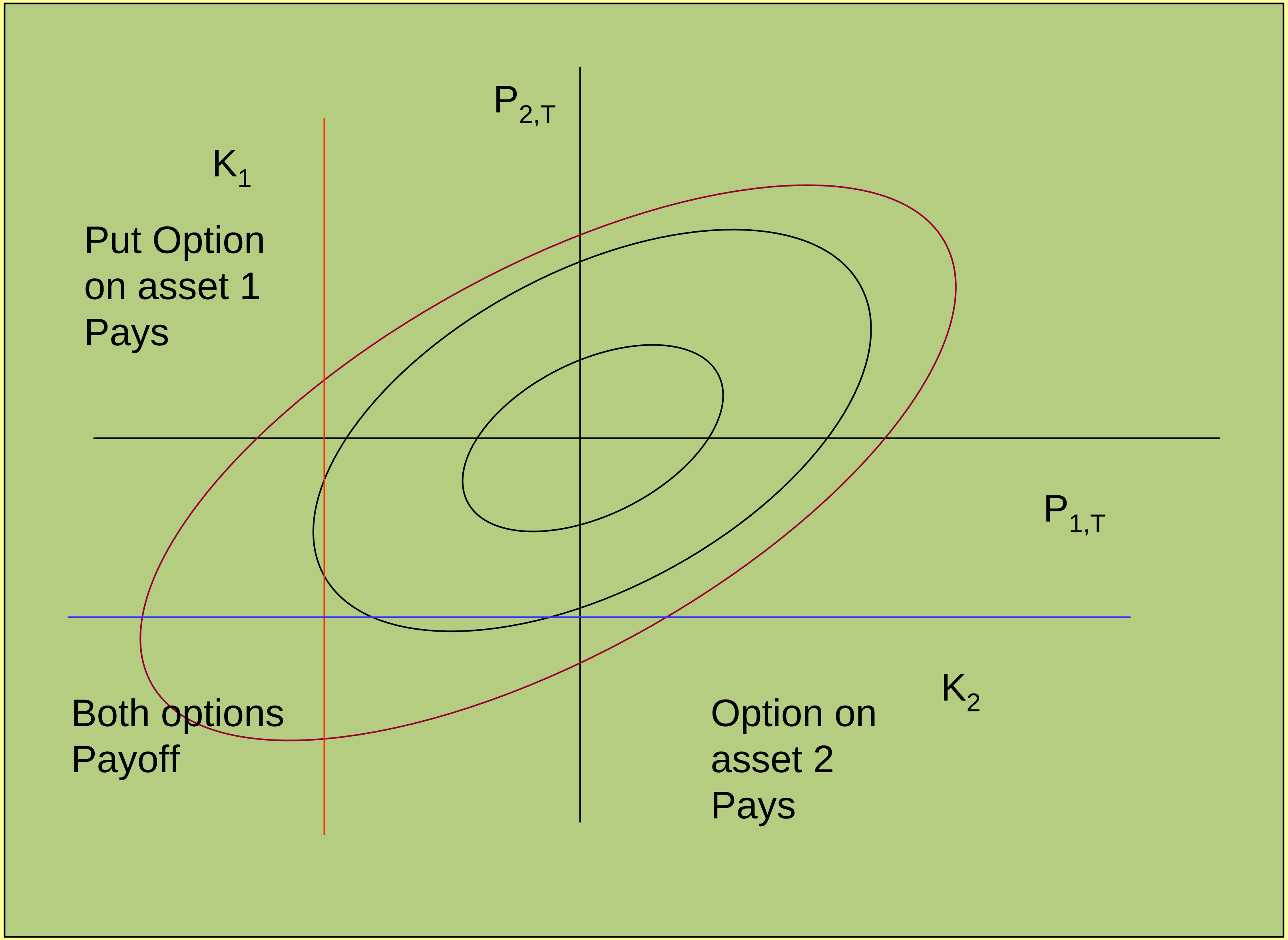

P2,T
K1
Put Option
on asset 1
Pays
P1,T
K2
Both options
Payoff
Option on
asset 2
Pays

Symmetric Tail Dependence



Lower Tail Dependence





$P_{2,T}$
K_1
Put Option
on asset 1
Pays
$P_{1,T}$
K_2
Both options
Payoff
Option on
asset 2
Pays

CONTAGION

- WHERE ARE MY CORRELATIONS WHEN I NEED THEM?
- WHEN COUNTRIES DECLINE TOGETHER MORE THAN CAN BE EXPECTED FROM THE NORMAL CORRELATION PATTERN, IT IS CALLED CONTAGION.
- CORRELATIONS AND VOLATILITIES APPEAR TO MOVE TOGETHER.

CREDIT DERIVATIVES

- IT IS WELL DOCUMENTED THAT THE MULTIVARIATE NORMAL DENSITY UNDERPRICES JOINT EXTREME EVENTS SUCH AS DEFAULTS.
- INDUSTRY HAS ADOPTED A T-COPULA TO PRICE CREDIT BASKETS and CDO’s.
- TAIL DEPENDENCE IS ESSENTIAL IN THESE MODELS.

THE PURPOSE OF MY TALK TODAY

TIME SERIES ANALYSIS OF DOWNSIDE RISK

PURPOSE OF MY TALK TODAY

- TO SHOW HOW DOWNSIDE RISK CAN BE MODELED AS A TIME SERIES PROCESS
- USING SIMPLY TIME AGGREGATION OF STANDARD TIME SERIES MODELS
- *CONSEQUENTLY*
- DOWNSIDE RISK CAN BE PREDICTED
- DYNAMIC HEDGING AND DYNAMIC PORTFOLIO STRATEGIES CAN BE IMPLEMENTED.

AN ECONOMETRIC FRAMEWORK

- MODEL THE ONE PERIOD RETURN AND CALCULATE THE MULTI-PERIOD CONDITIONAL DISTRIBUTION
- RETURN FROM t UNTIL t + T IS:

$$R_t^T = \sum_{j=t+1}^{T+t} r_j$$

- THE DISTRIBUTION CONDITIONAL ON TODAY'S INFORMATION IS:

$$R_t^T \,\big|\, \mathrm{F}_t \sim f_t^T\left(R_t^T\right)$$

ALL MEASURES CAN BE DERIVED FROM THE ONE PERIOD DENSITY

- EVALUATE ANY MEASURE BY REPEATEDLY SIMULATING FROM THE ONE PERIOD CONDITIONAL DISTRIBUTION:

$$f_t\left(r_{t+1}\right)$$

- METHOD:
  - Draw r_{t+1}
  - Update density and draw observation t+2
  - Continue until T returns are computed.
  - Compute measure of downside risk

A MODEL

- $$r_t = \sqrt{h_t}\varepsilon_t, \;\; \varepsilon_t \sim i.i.d.$$
  $$E_{t-1}(r_t) = 0, h_t = V_{t-1}(r_t)$$
- ASYMMETRY FOLLOWS FROM ASYMMETRY IN EPSILON
- HOWEVER FOR MULTI-PERIOD RETURNS, THERE IS ANOTHER SOURCE – ASYMMETRIC VOLATILITY.

The ARCH Model

- The ARCH model of Engle(1982) is a family of specifications for the conditional variance.
- The q^{th} order ARCH or ARCH(q) model is

$$h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}{}^2$$

- Notice that it is a simple generalization of both constant variance and rolling variance estimates called “historical volatilities”.

GARCH

- The Generalized ARCH model of Bollerslev(1986) is an ARMA version of this model. GARCH(1,1) is a weighted average of three volatility forecasts:

$$h_t = \omega + \alpha r_{t-1}{}^2 + \beta h_{t-1}$$

Asymmetric Volatility

- Often negative shocks have a bigger effect on volatility than positive shocks.
- Nelson(1987) introduced the EGARCH model to incorporate this effect.
- I will use a Threshold GARCH or TARCH

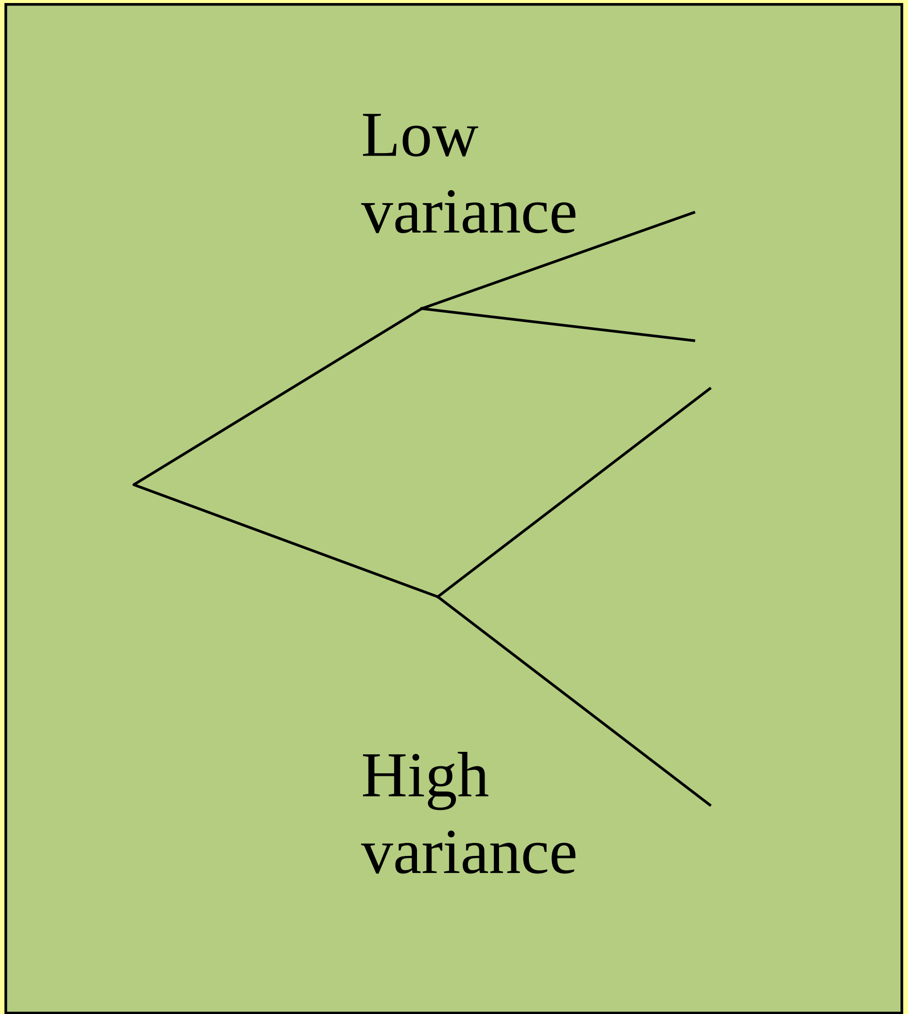
$$h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 I_{(r_{t-1}<0)} + \beta h_{t-1}$$

NEW ARCH MODELS

- GJR-GARCH
- TARCH
- STARCH
- AARCH
- NARCH
- MARCH
- SWARCH
- SNPARCH
- APARCH
- TAYLOR-SCHWERT
- FIGARCH
- FIEGARCH
- Component
- Asymmetric Component
- SQGARCH
- CESGARCH
- Student t
- GED
- SPARCH
- Autoregressive Conditional Density
- Autoregressive Conditional Skewness

TWO PERIOD RETURNS

- Two period return is the sum of two one period continuously compounded returns
- Look at binomial tree version
- Asymmetric Volatility gives negative skewness



ANALYTICALLY: TARCH
WITH SYMMETRIC INNOVATIONS

$$E\left(r_t + r_{t+1}\right)^3 = E\left(r_t^3 + 3r_t^2 r_{t+1} + 3r_t r_{t+1}^2 + r_{t+1}^3\right)$$

$$= 0 + 0 + 3E\left(r_t h_{t+1}\right) + 0$$

$$= 3E\left[r_t\left(\omega + \alpha r_t^2 + \gamma r_t^2 I_{(r_t<0)} + \beta h_t\right)\right]$$

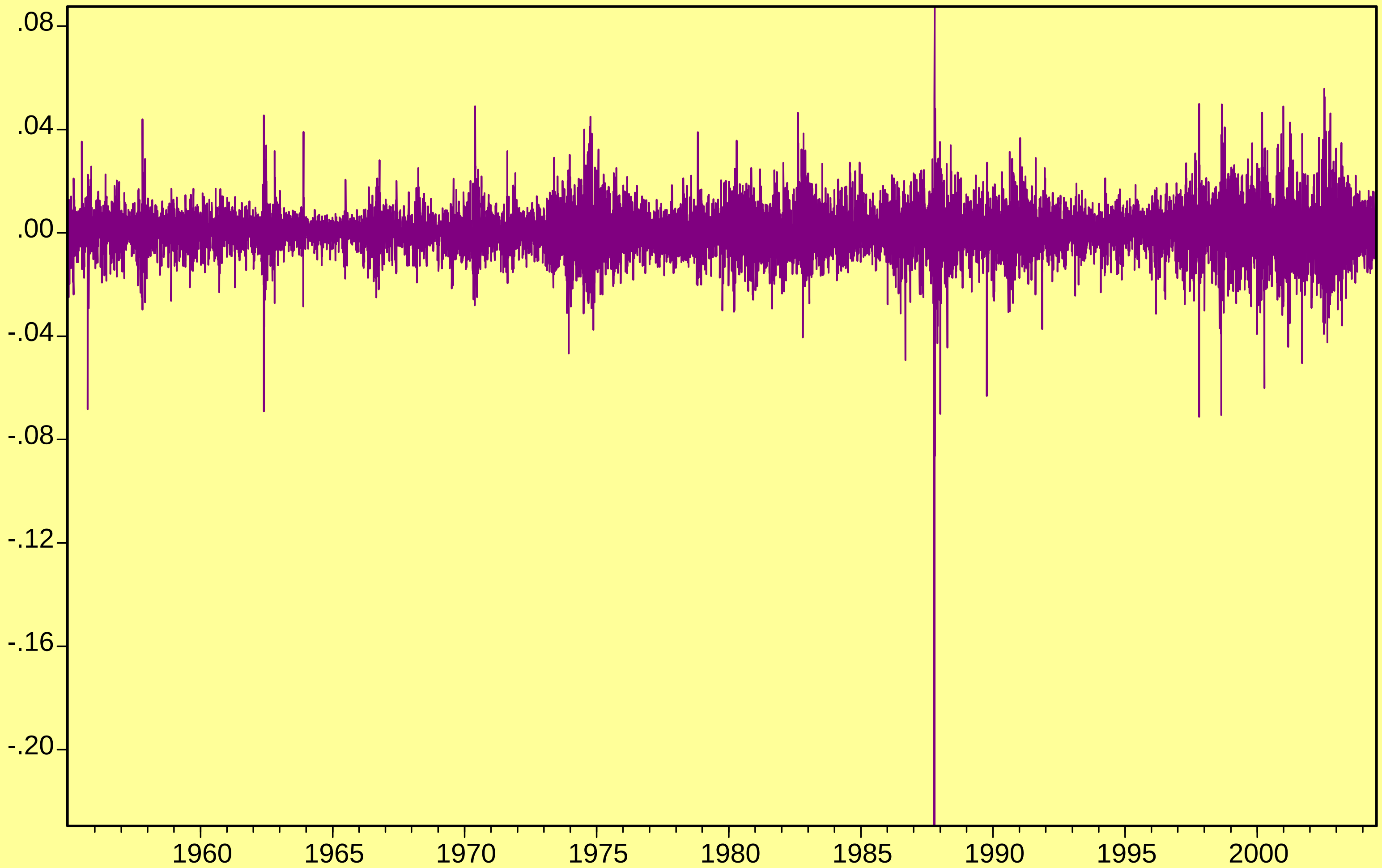
$$= 3\gamma E\left(r_t^3 I_{(r_t<0)}\right) < 0$$

and the conditional third moment is

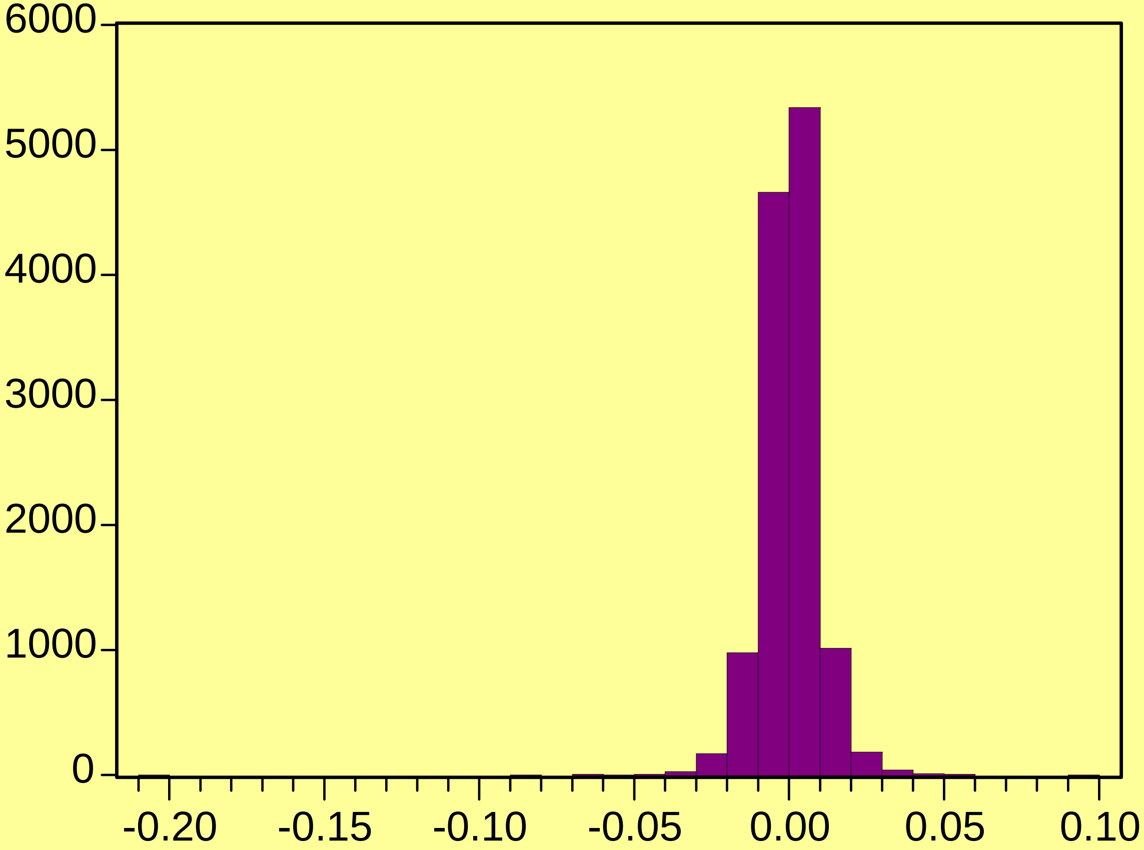
$$E_{t-1}\left(r_t + r_{t+1}\right)^3 = 3\gamma E_{t-1}\left(r_t^3 I_{(r_t<0)}\right) < 0$$

STYLIZED FACTS

S&P 500 DAILY RETURNS



HISTOGRAM OF S&P500 DAILY RETURNS

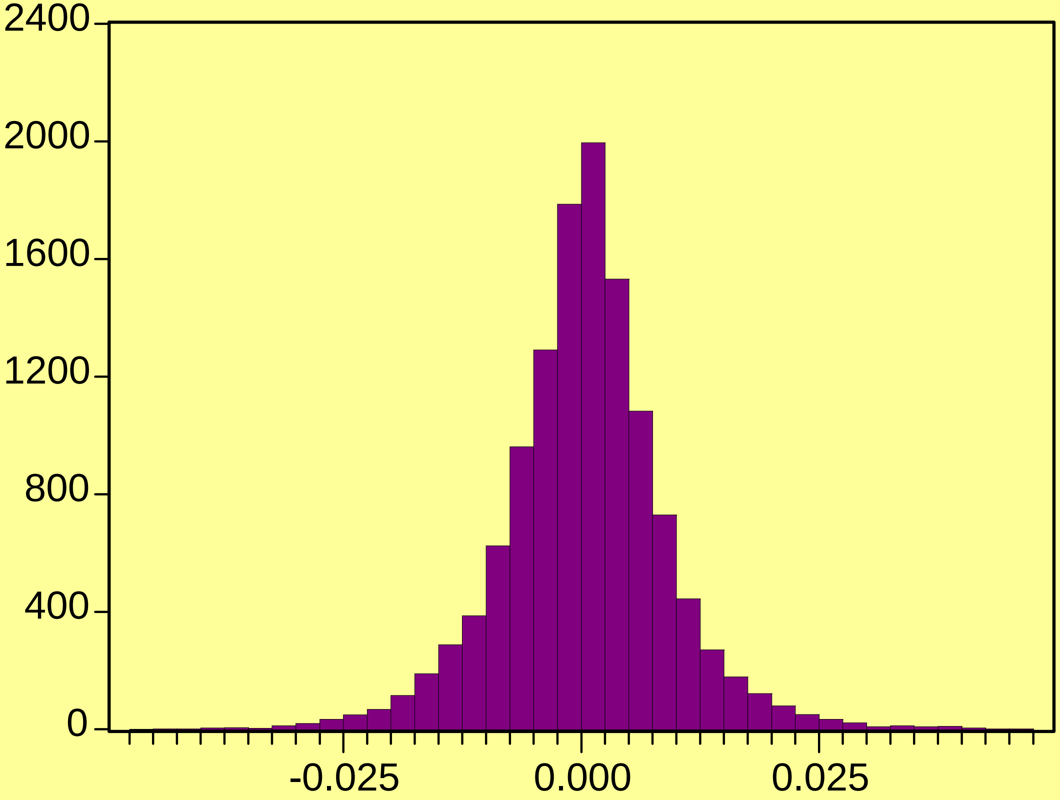


Series: RETUSSP
Sample 1/4/1955 TO 6/25/2004
Observations 12455

Mean	0.000318
Median	0.000375
Maximum	0.090994
Minimum	-0.204669
Std. Dev.	0.009179
Skewness	-0.926286
Kurtosis	28.00273
Jarque-Bera	326200.8
Probability	0.000000

TRIMMING .001 IN EACH TAIL

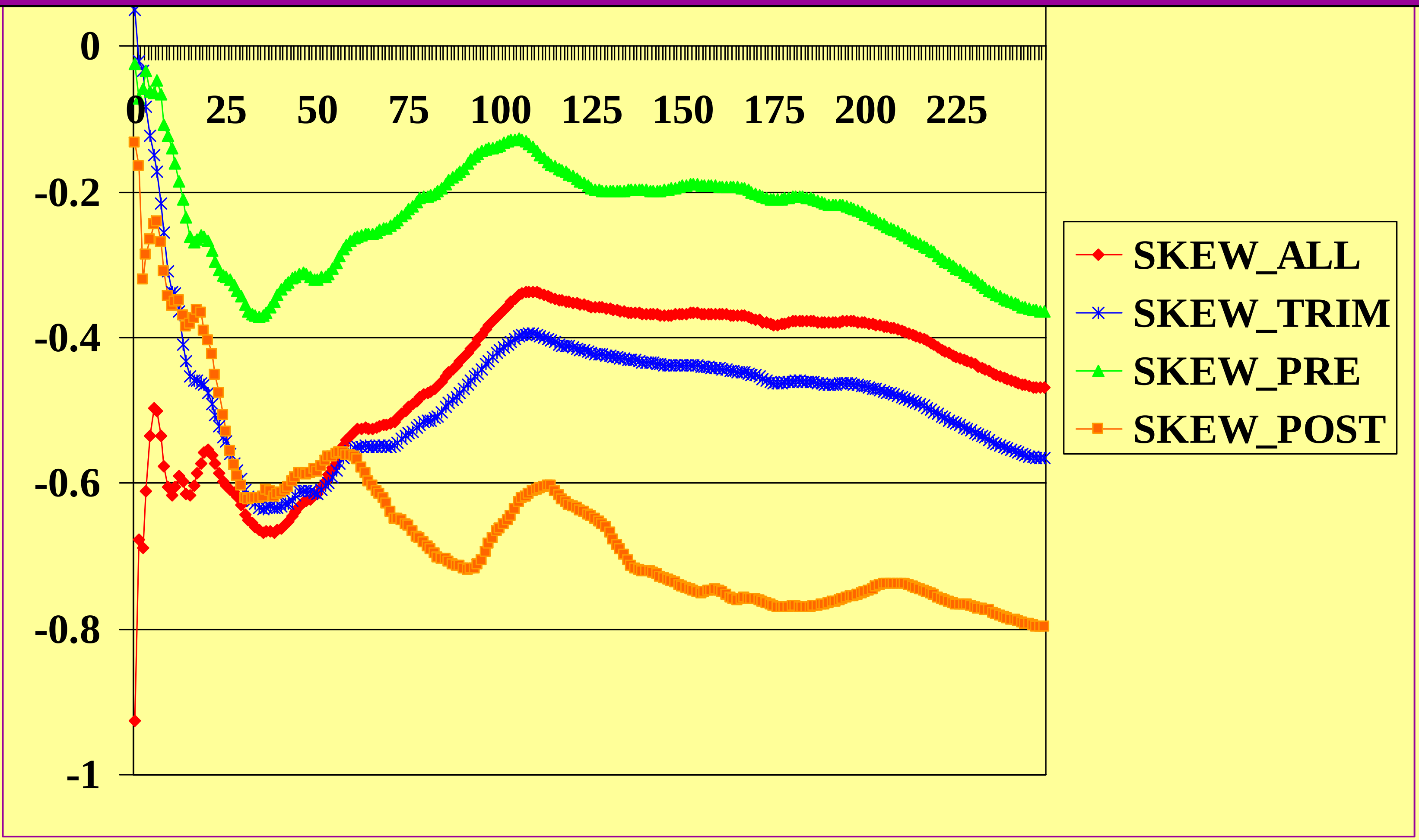
(8 DAYS)



Series: RETUSSP
Sample 1 12455 IF Y>LOWTRIM AND Y<HIGHTRIM
Observations 12431

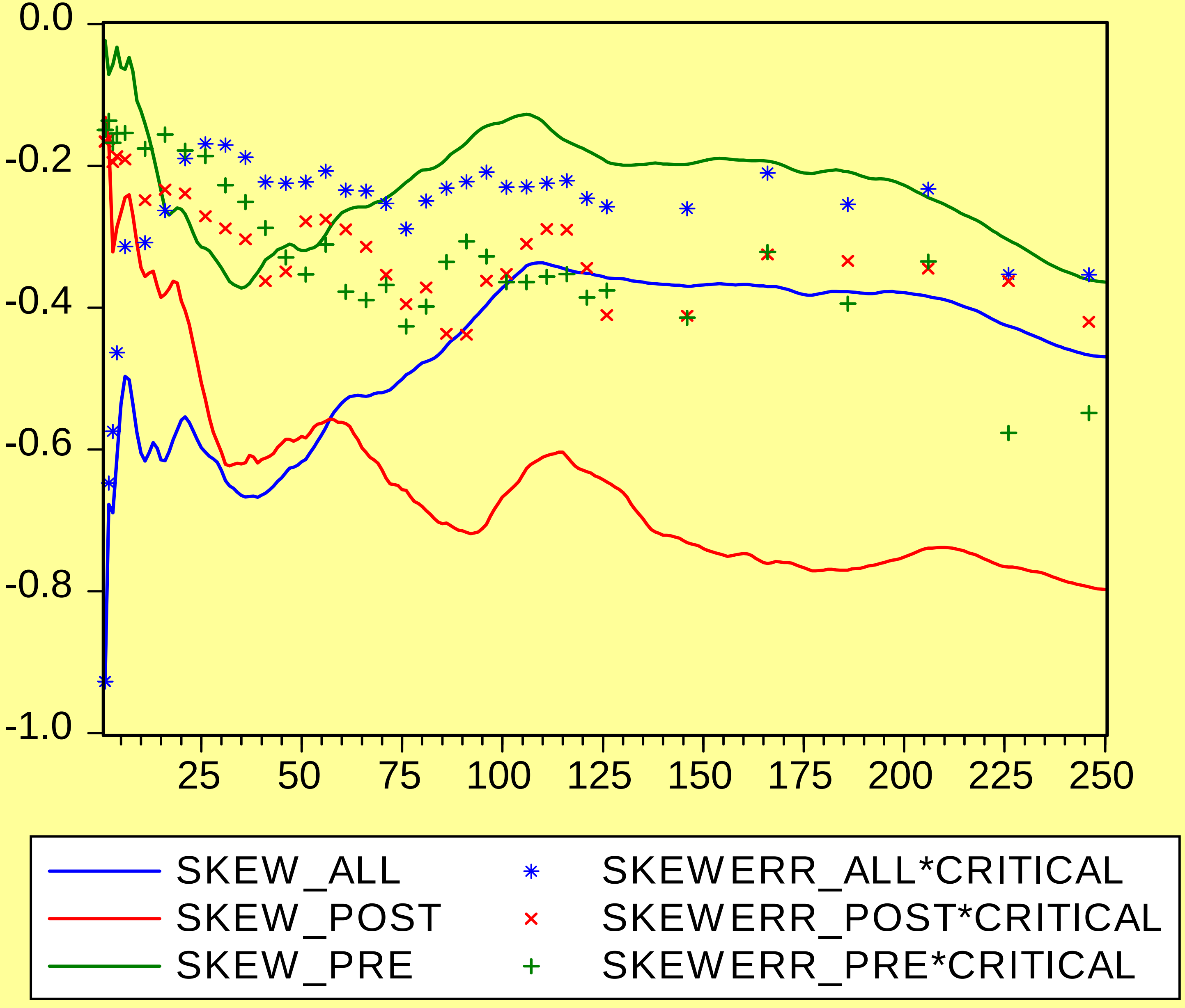
Mean	0.000338
Median	0.000375
Maximum	0.046486
Minimum	-0.045594
Std. Dev.	0.008633
Skewness	0.049617
Kurtosis	5.380668
Jarque-Bera	2940.670
Probability	0.000000

SKEWNESS OF MULTIPERIOD RETURNS



STANDARD ERRORS

- ARE THESE DIFFERENCES SIGNIFICANT?
- THE INFERENCE IS COMPLICATED BY THE OVERLAPPING OBSERVATIONS AND BY THE DEPENDENCE DUE TO ESTIMATING THE MEAN.
- FROM SIMPLE ROBUST TESTS, SIZE CORRECTED BY MONTE CARLO, THESE ARE SIGNIFICANT.

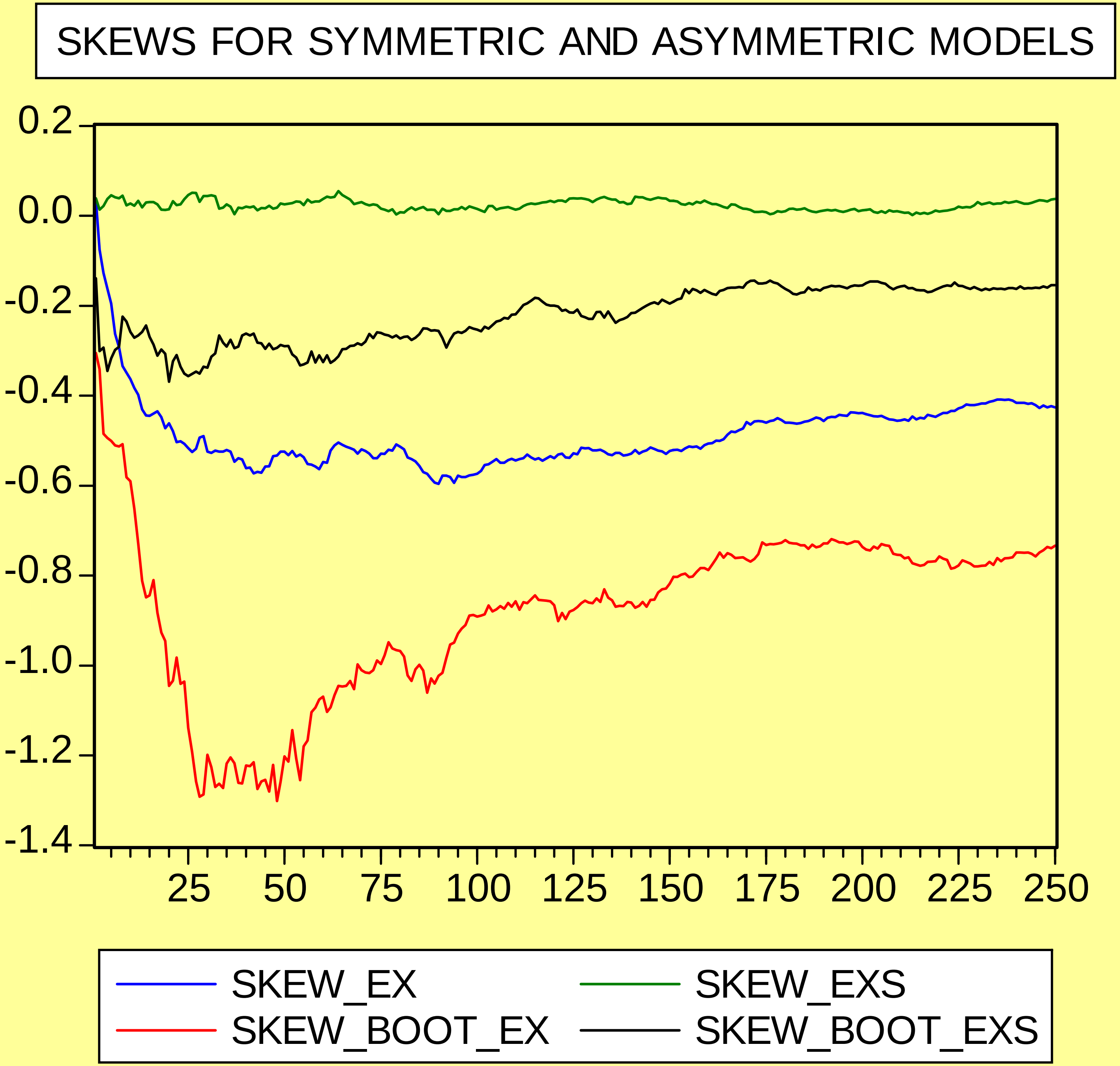

0.0
-0.2
-0.4
-0.6
-0.8
-1.0
25
50
75
100
125
150
175
200
225
250
SKEW_ALL
SKEW_POST
SKEW_PRE
SKEWERR_ALL*CRITICAL
SKEWERR_POST*CRITICAL
SKEWERR_PRE*CRITICAL

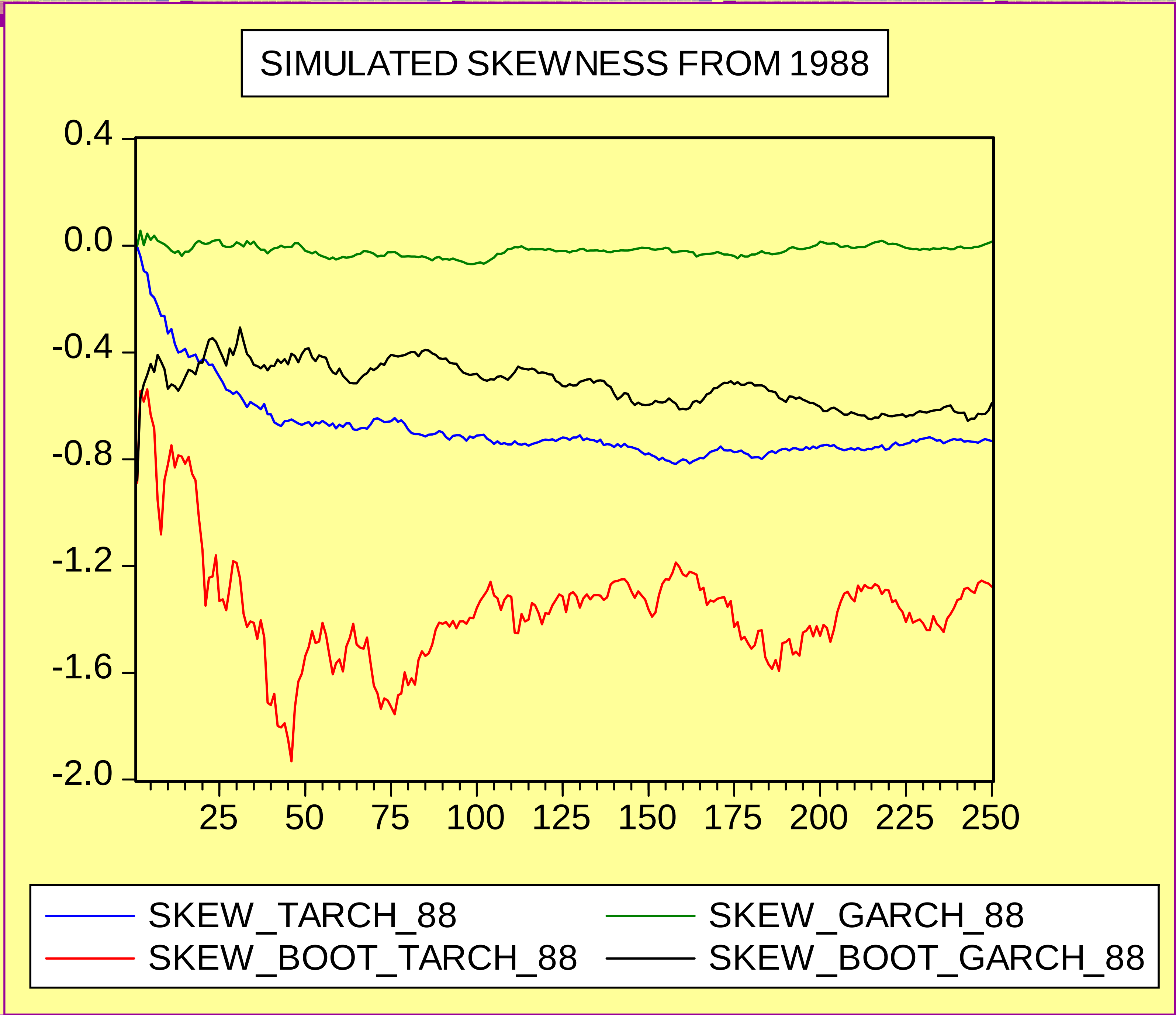
EVIDENCE FROM DERIVATIVES

- THE HIGH PRICE OF OUT-OF-THE-MONEY EQUITY PUT OPTIONS IS WELL DOCUMENTED
- THIS IMPLIES SKEWNESS IN THE RISK NEUTRAL DISTRIBUTION
- MUCH OF THIS IS PROBABLY DUE TO SKEWNESS IN THE EMPIRICAL DISTRIBUTION OF RETURNS.
- DATA MATCHES EVIDENCE THAT THE OPTION SKEW IS ONLY POST 1987.

MATCHING THE STYLIZED FACTS

- ESTIMATE DAILY MODEL
- SIMULATE 250 CUMULATIVE RETURNS 10,000 TIMES WITH SEVERAL DATA GENERATING PROCESSES
- CALCULATE SKEWNESS AT EACH HORIZON


SKEWS FOR SYMMETRIC AND ASYMMETRIC MODELS
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
25
50
75
100
125
150
175
200
225
250
SKEW_EX
SKEW_EXS
SKEW_BOOT_EX
SKEW_BOOT_EXS


SIMULATED SKEWNESS FROM 1988
0.4
0.0
-0.4
-0.8
-1.2
-1.6
-2.0
25
50
75
100
125
150
175
200
225
250
SKEW_TARCH_88
SKEW_GARCH_88
SKEW_BOOT_TARCH_88
SKEW_BOOT_GARCH_88

IMPLICATIONS

- Multi-period empirical returns are more skewed than one period returns (omitting 1987 crash)
- Asymmetric volatility is needed to explain this.
- Skewness has increased since 1987, particularly for longer horizons.
- Simulated skewness is noisy because higher moments do not exist when the persistence is so close to one. Presumably this is true for the data too.
- Many other asymmetric models could be compared on this basis.

MULTIVARIATE MODELS

DOWNSIDE RISK RESULTS FROM TIME AGGREGATION WITH:

- ASYMMETRIC CORRELATIONS
  - CORRELATIONS RISE PARTICULARLY AFTER TWO ASSETS BOTH DECLINE. (Asymmetric DCC (Cappiello, Engle, Sheppard(2004))
- VOLATILITY SHOCKS ARE CORRELATED
  - PURE VARIANCE COMMON FEATURES(Engle, Marcucci(2005))
  - FACTOR MODELS (Engle Ng and Rothschild(1992))
  - CREDIT RISK MODEL(Engle, Berd, Voronov(2005))

FACTOR ARCH

- RETURNS ARE DRIVEN BY A SMALL NUMBER OF FACTOR SHOCKS, f_t .
- FACTORS DRIVE VOLATILITIES AND CORRELATIONS

$$r_t = Bf_t + u_t$$

$$V_{t-1}\left(r_t\right) = B\Omega_t B + D_t$$

$$V_{t-1}\left(f_t\right) = \Omega_t, V_{t-1}\left(u_t\right) = D_t$$

DOWNSIDE RISK IN THE CAPM

- The return on a stock can be decomposed into systematic and idiosyncratic returns using the beta of the stock
- $$r_{i,t} = \beta_i r_{m,t} + \varepsilon_{i,t}$$
- If the market declines substantially, many stocks will decline. There will be skewness in each stock and downside risk in the portfolio.

SKEWNESS

- Under the standard assumptions, the skewness of return *i* is related to the return of the market by $s_i = s_m R^3$ where R^3 is the conventional R^2 raised to the 3/2 power.
- Notice that all stocks will then have skewness but that it will be less than for the market.

TAIL DEPENDENCE

- The probability that two stocks will both underperform some threshold can be calculated conditional on the market return.

$$P\left(r_i < k \ and \ r_j < k\right) = E\left(P\left(r_i < k \ and \ r_j < k \middle| r_m\right)\right)$$

$$= E\left(P\left(\varepsilon_i < k - \beta_i r_m \middle| r_m\right) P\left(\varepsilon_j < k - \beta_j r_m \middle| r_m\right)\right)$$

$$= P\left(r_i < k\right) P\left(r_j < k\right)$$

$$+ Cov\left(P\left(\varepsilon_i < k - \beta_i r_m \middle| r_m\right), P\left(\varepsilon_j < k - \beta_j r_m \middle| r_m\right)\right)$$

$$= E\left(\Phi\left(k - \beta_i r_m\right)\Phi\left(k - \beta_j r_m\right)\right) \text{ under normality}$$

SUMMARY

- ASYMMETRIC VOLATILITY IN THE MARKET FACTOR IMPLIES
  - SKEWNESS IN MULTIPERIOD MARKET RETURNS
  - SKEWNESS IN MULTIPERIOD EQUITY RETURNS
  - LOWER TAIL DEPENDENCE IN EQUITY RETURNS

IMPLICATIONS FOR FINANCIAL MANAGEMENT

IMPLICATIONS FOR RISK MANAGEMENT

- MULTI-PERIOD RISKS MAY BE SUBSTANTIALLY DIFFERENT FROM ONE PERIOD RISKS.
- THE MULTI-PERIOD RISK CHANGES OVER TIME AND CAN BE FORECAST.
- BIG MARKET DECLINES ARE MORE LIKELY WHEN VOLATILITY IS HIGH

IMPLICATIONS FOR DERIVATIVE HEDGING

- AS EACH NEW PERIOD RETURN IS OBSERVED, THE DERIVATIVE CAN BE REPRICED AND THE HEDGE UPDATED.
- GREEKS CAN BE CALCULATED FROM SIMULATION PRICING TO SIMPLIFY THE UPDATING

IMPLICATIONS FOR PORTFOLIO SELECTION

- MEAN VARIANCE PORTFOLIO OPTIMIZATION WILL MISS THESE ASYMMETRIES.

- HIGH FREQUENCY REBALANCING WILL GIVE *EARLY WARNING* OF DOWNSIDE RISK.

HOW TO DO THIS?

- SUBOPTIMAL METHOD 1
  - MYOPIC ASSET ALLOCATION ON A HIGH FREQUENCY BASIS.
  - AS VOLATILITIES RISE, YOU NATURALLY SHIFT OUT OF RISKY ASSETS.
- SUBOPTIMAL METHOD 2
  - MULTI-PERIOD FORECAST OF RISK GIVES AN EX-ANTE OPTIMAL PLAN.
  - OVERINVEST WHEN VOLATILITY IS LOW AND UNDERINVEST WHEN IT IS HIGH

OPTIMAL METHOD

- DYNAMIC PROGRAMMING:
  - WHEN VOLATILITY IS LOW, UNDERINVEST, RECOGNIZING THAT THIS PLAN MAY CHANGE WHEN THE SUBSEQUENT VOLATILITY IS OBSERVED
  - SEE COLACITO AND ENGLE(2004)

EXPECTED RETURNS

- EACH OF THESE METHODS REQUIRES EXPECTED RETURNS.
- THE LISTED IMPLICATIONS ARE BASED ON THE ASSUMPTION THAT EXPECTED RETURNS ARE UNCHANGED.
- IS THIS REASONABLE?

BUT IF EVERYBODY DID THIS?

- IF ALL AGENTS FOLLOW THIS PATTERN THEN EXPECTED RETURNS WOULD NECESSARILY ADJUST. RETURNS WOULD INSTANTANEOUSLY MOVE ENOUGH TO RESTORE EQUILIBRIUM. CAMPBELL AND HENTSCHEL(1992)
- IN A REPRESENTATIVE AGENT WORLD, THERE WOULD NO LONGER BE A MOTIVE FOR ADJUSING TO CHANGES IN RISK.
- CHANGES IN RISK WOULD LEAD TO UNAVOIDABLE CAPITAL GAINS OR LOSSES.
- DERIVATIVE REPLICATION STRATEGIES WOULD CONTINUE TO BE USEFUL.

HOWEVER

- THERE IS NO REASON TO BELIEVE DOWNSIDE RISK WOULD DISAPPEAR OR COLLAPSE TO AN INSTANT IN TIME.
- WITH HETEROGENEITY, THERE WOULD STILL BE REASONS TO REBALANCE.
- FROM A MICROSTRUCTURE POINT OF VIEW IT IS DIFFICULT TO IMAGINE HOW THE PRICES COULD INSTANTANEOUSLY ADJUST TO VOLATILITY NEWS.
- EXPECTED RETURNS WOULD BE EXCEEDINGLY DIFFICULT TO ESTIMATE AT THIS HIGH FREQUENCY
- MAYBE WE ARE ALREADY AT THIS POINT SO THAT DOWNSIDE RISK IS FULLY AND INSTANTLY PRICED.

CONCLUSIONS

- ASYMMETRIC VOLATILITY AND CORRELATION MODELS ARE POWERFUL TOOLS FOR ANALYZING DOWNSIDE RISK
- ONE PERIOD MODELS HAVE BIG IMPLICATIONS ABOUT LONG HORIZON OF RETURNS
- THE UPDATING OF VOLATILITY AND RISK MEASURES HAS A NATURAL APPLICATION TO DERIVATIVE HEDGING AND POSSIBLY PORTFOLIO REBALANCING.